Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number	Candidate Number

Time 1 hour 40 minutes

Paper reference **8FM0/01**

Further Mathematics

Advanced Subsidiary
PAPER 1: Core Pure Mathematics

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 80.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- Good luck with your examination.

Turn over

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Pearson

1. $\mathbf{P} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

(a) (i) Describe fully the single geometrical transformation P represented by the matrix **P**.

(ii) Describe fully the single geometrical transformation Q represented by the matrix **Q**.

(4)

The transformation P followed by the transformation Q is the transformation R, which is represented by the matrix **R**.

(b) Determine **R**.

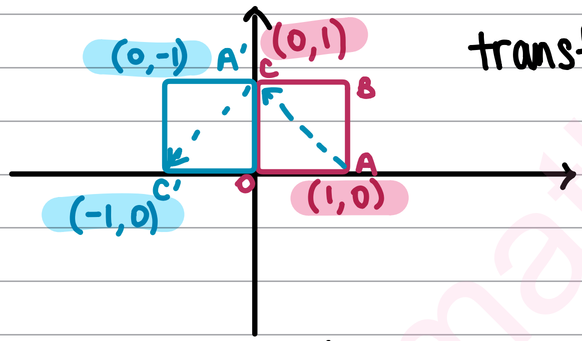
(1)

(c) (i) Evaluate the determinant of **R**.

(ii) Explain how the value obtained in (c)(i) relates to the transformation R.

(2)

1.a)(i) METHOD 1 (draw out transformation on unit square):



transformation by P: unit square has corners $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ → A, C

$PX = X'$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

A' C'

As can be seen from the diagram, the unit square is rotated 90° anticlockwise about the origin to form the blue image

$\therefore$ P is a 90° anticlockwise rotation about the origin

METHOD 2 (use formula booklet):

Formula booklet for anticlockwise rotation through θ about origin : $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

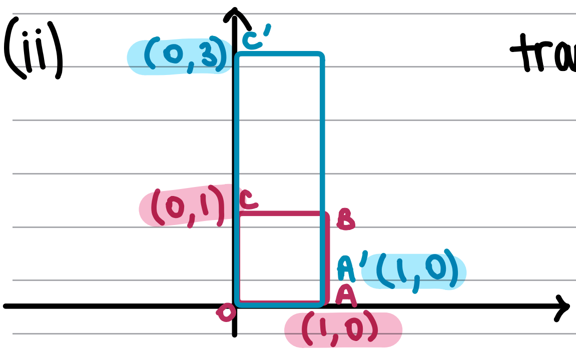
Compare to P $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\cos\theta = 0$

$\sin\theta = 1$ $\therefore \theta = 90°$

Question 1 continued

(ii)



transformation by Q:

$QX = X'$

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

A C (unit square) A' C'

As can be seen from the diagram, the unit square is stretched by a factor of 3 parallel to the y-axis to form the image

$\therefore$ Q is a stretch, scale factor 3 parallel to the y-axis

b) The transformation R is P followed by Q

Transformation A followed by B is represented by the matrix BA

$\therefore R = QP$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

MATRIX MULTIPLICATION:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

$$= \begin{bmatrix} 1(0)+0(1) & 1(-1)+0(0) \\ 0(0)+3(1) & 0(-1)+3(0) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}$$

Question 1 continued

c)(i) Determinant of $R\begin{bmatrix}0 & -1\\3 & 0\end{bmatrix}$

2×2 MATRIX DETERMINANT

$\left|\begin{bmatrix}a & b\\c & d\end{bmatrix}\right| = ad - bc$

$\det R = |R| = (0)(0) - (-1)(3)$

$= -(-3) = 3$

$|R| = 3$

(ii) Because matrix R is: → rotation (where the area of the object doesn't change)

→ stretch scale factor 3 (where the area increases by scale factor 3)

∴ the overall area scale factor change would be ×3

The determinant of a transformation matrix gives the area scale factor for the transformation ∴ it is 3

(Total for Question 1 is 7 marks)

P 6 6 7 9 0 A 0 3 3 2

Turn over

2. The cubic equation

$$9x^3 - 5x^2 + 4x + 7 = 0$$

has roots α, β and γ.

Without solving the equation, find the cubic equation whose roots are $(3\alpha - 2)$, $(3\beta - 2)$ and $(3\gamma - 2)$, giving your answer in the form $aw^3 + bw^2 + cw + d = 0$, where a, b, c and d are integers to be determined.

(5)

2. METHOD 1 (using substitution):

$9x^3 - 5x^2 + 4x + 7 = 0 \leftarrow$ roots α, β, γ

$w = 3x - 2$

$x = \frac{w+2}{3}$

$9\left(\frac{w+2}{3}\right)^3 - 5\left(\frac{w+2}{3}\right)^2 + 4\left(\frac{w+2}{3}\right) + 7 = 0$

$9\left(\frac{(w^2+4w+4)(w+2)}{27}\right) - 5\left(\frac{w^2+4w+4}{9}\right) + 4\left(\frac{w+2}{3}\right) + 7 = 0$

$\frac{w^3+6w^2+12w+8}{3} - \left(\frac{5w^2+20w+20}{9}\right) + \frac{4w+8}{3} + 7 = 0$

(multiply both sides by 9)

$\times 9$

$3w^3 + 18w^2 + 36w + 24 - 5w^2 - 20w - 20 + 12w + 24 + 63 = 0$

$3w^3 + 13w^2 + 28w + 91 = 0$

METHOD 2 (using sum/product rules for polynomials):

$ax^3 + bx^2 + cx + d = 0$

$\alpha+\beta+\gamma = \frac{-b}{a}$ $\quad \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$ $\quad \alpha\beta\gamma = \frac{-d}{a}$

P 6 6 7 9 0 A 0 4 3 2

Question 2 continued

$\alpha+\beta+\gamma = \frac{5}{9}$ $\quad\alpha\beta+\alpha\gamma+\beta\gamma = \frac{4}{9}$ $\quad\alpha\beta\gamma = -\frac{7}{9}$

$\underset{p}{(3\alpha-2)}, \underset{q}{(3\beta-2)}, \underset{r}{(3\gamma-2)}$: new roots

↳ new equation

ax^3+bx^2+cx+d

let $a=1$

$$\frac{-b}{a} = p+q+r$$
$$= 3\alpha-2+3\beta-2+3\gamma-2$$
$$= p+q+r$$
$$= 3(\alpha+\beta+\gamma)-6$$
$$= 3\left(\frac{5}{9}\right)-6 = -\frac{13}{3} \qquad b = \frac{13}{3}$$

$$\frac{c}{a} = pq+pr+qr$$
$$= 9\alpha\beta-6\alpha-6\beta+4+9\alpha\gamma-6\alpha-6\gamma+4+9\beta\gamma-6\beta-6\gamma+4$$
$$= 9\alpha\beta+9\alpha\gamma+9\beta\gamma-12\alpha-12\beta-12\gamma+12$$
$$= 9(\alpha\beta+\alpha\gamma+\beta\gamma)-12(\alpha+\beta+\gamma)+12$$
$$= 9\left(\frac{4}{9}\right)-12\left(\frac{5}{9}\right)+12 = \frac{28}{3} \qquad c = \frac{28}{3}$$

$$\frac{-d}{a} = pqr = pq(r$$
$$= (9\alpha\beta-6\alpha-6\beta+4)(3\gamma-2)$$
$$= 27\alpha\beta\gamma-18\alpha\beta-18\alpha\gamma+12\alpha-18\beta\gamma+12\beta+12\gamma-8$$
$$= 27(\alpha\beta\gamma)-18(\alpha\beta+\alpha\gamma+\beta\gamma)+12(\alpha+\beta+\gamma)-8$$
$$= 27\left(-\frac{7}{9}\right)-18\left(\frac{4}{9}\right)+12\left(\frac{5}{9}\right)-8 = -\frac{91}{3}$$

(Total for Question 2 is 5 marks)

Question 2 continued

$d = \frac{91}{3}$

$x^3 + \frac{13}{3}x^2 + \frac{28}{3}x + \frac{91}{3} = 0$ ← To get integer co-efficients multiply by 3

$3x^3 + 13x^2 + 28x + 91 = 0$

(Total for Question 2 is 5 marks)

3. (a) Use the standard results for summations to show that for all positive integers n

$$\sum_{r=1}^{n}(5r-2)^2 = \frac{1}{6}n(an^2+bn+c)$$

where a, b and c are integers to be determined.

(5)

(b) Hence determine the value of k for which

$$\sum_{r=1}^{k}(5r-2)^2 = 94k^2$$

(4)

3.a) $(5r-2)^2 = 25r^2 - 20r + 4$

$$\sum_{r=1}^{n}(25r^2 - 20r + 4)$$

$$= 25\sum_{r=1}^{n} r^2 - 20\sum_{r=1}^{n} r + 4\sum_{r=1}^{n}(1)$$

USING STANDARD SUMMATIONS

$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ from formulae booklet

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ triangular no. formulae

$$= \frac{25}{6}n(n+1)(2n+1) - \frac{20}{2}n(n+1) + 4n$$

$$= \frac{n}{6}\left(25(n+1)(2n+1) - 60(n+1) + 24\right)$$

$$= \frac{n}{6}\left(25(2n^2+3n+1) - 60n - 60 + 24\right)$$

$$= \frac{n}{6}\left(50n^2 + 75n + 25 - 60n - 36\right)$$

$$= \frac{n}{6}\left(50n^2 + 15n - 11\right)$$

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P 6 6 7 9 0 A 0 6 3 2

Question 3 continued

b) $n = k$

$$\sum_{r=1}^{k}(5r-2)^2 = \frac{k}{6}\left(50k^2+15k-11\right) = 94k^2$$

$$\cancel{k}\left(50k^2+15k-11\right) = 564k^{\cancel{2}}$$

$$50k^2+15k-11 = 564k$$

$$50k^2-549k-11=0$$

$$(k-11)(50k+1)=0$$

$k=11$ U $\cancel{-\frac{1}{50}}$ ← reject negative non-integer solution

$\therefore k=11$

(Total for Question 3 is 9 marks)

4.
$$\mathbf{M} = \begin{pmatrix} 2 & 1 & 4 \\ k & 2 & -2 \\ 4 & 1 & -2 \end{pmatrix} \qquad \mathbf{N} = \begin{pmatrix} k-7 & 6 & -10 \\ 2 & -20 & 24 \\ -3 & 2 & -1 \end{pmatrix}$$

where k is a constant.

(a) Determine, in simplest form in terms of k, the matrix $\mathbf{MN}$. **(2)**

(b) Given that $k = 5$

(i) write down $\mathbf{MN}$

(ii) hence write down $\mathbf{M}^{-1}$ **(2)**

(c) Solve the simultaneous equations

$$\begin{aligned} 2x + y + 4z &= 2 \\ 5x + 2y - 2z &= 3 \\ 4x + y - 2z &= -1 \end{aligned}$$

(2)

(d) Interpret the answer to part (c) geometrically. **(1)**

4. a) MATRIX MULTIPLICATION (3x3)

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} (aj+bm+cp) & (ak+bn+cq) & (al+bo+cr) \\ (dj+em+fp) & (dk+en+fq) & (dl+eo+fr) \\ (gj+hm+ip) & (gk+hn+iq) & (gl+ho+ir) \end{bmatrix}$$

$$MN = \begin{bmatrix} 2 & 1 & 4 \\ k & 2 & -2 \\ 4 & 1 & -2 \end{bmatrix} \begin{bmatrix} k-7 & 6 & -10 \\ 2 & -20 & 24 \\ -3 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2(k-7)+1(2)+4(-3) & 2(6)+1(-20)+4(2) & 2(-10)+1(24)+4(-1) \\ k(k-7)+2(2)-2(-3) & k(6)+2(-20)-2(2) & k(-10)+2(24)-2(-1) \\ 4(k-7)+1(2)-2(-3) & 4(6)+1(-20)-2(2) & 4(-10)+1(24)-2(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 2k-24 & 0 & 0 \\ k^2-7k+10 & 6k-44 & -10k+50 \\ 4k-20 & 0 & -14 \end{bmatrix}$$

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Question 4 continued

b) (i) $MN = \begin{bmatrix} 2k-24 & 0 & 0 \\ k^2-7k+10 & 6k-44 & -10k+50 \\ 4k-20 & 0 & -14 \end{bmatrix}$

if $k=5$

$MN = \begin{bmatrix} 2(5)-24 & 0 & 0 \\ (5)^2-7(5)+10 & 6(5)-44 & -10(5)+50 \\ 4(5)-20 & 0 & -14 \end{bmatrix}$

$= \begin{bmatrix} -14 & 0 & 0 \\ 0 & -14 & 0 \\ 0 & 0 & -14 \end{bmatrix}$

(ii) $MN = -14\,I$

I 3x3 identity matrix $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

matrix multiplied by its inverse gives identity

$X \times X^{-1} = I$

inverse X

$\therefore M \times M^{-1} = I$

if $M \times N = -14\,I$

then $N = -14M^{-1}$

$M^{-1} = -\frac{1}{14}N = -\frac{1}{14}\begin{bmatrix} -2 & 6 & -10 \\ 2 & -20 & 24 \\ -3 & 2 & -1 \end{bmatrix}$ (when $k=5$)

Question 4 continued

c) $\begin{bmatrix} 2 & 1 & 4 \\ 5 & 2 & -2 \\ 4 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

multiply both sides by M^{-1} $\quad M \ X = B$

$M^{-1}M\ X = M^{-1}B$

$M \times M^{-1} = I$ (Identity)

$IX = M^{-1}B$

$X = M^{-1}B = -\frac{1}{14} N B$

$= -\frac{1}{14}\begin{bmatrix} -2 & 6 & -10 \\ 2 & -20 & 24 \\ -3 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{14}\begin{bmatrix} 24 \\ -80 \\ 1 \end{bmatrix}$

$x = -\frac{12}{7} \quad y = \frac{40}{7} \quad z = -\frac{1}{14}$

d) The matrix X gives the only point at which the 3 planes (from the equations in (c)) meet at.

5.

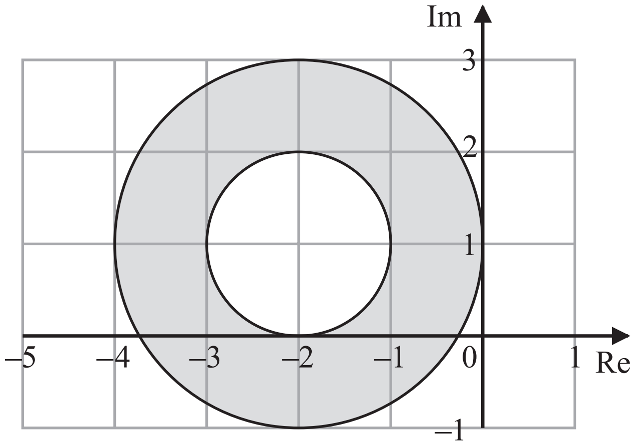


Figure 1

Figure 1 shows an Argand diagram.

The set P, of points that lie within the shaded region including its boundaries, is defined by

$$P = \{z \in \mathbb{C} : a \leqslant |z + b + c\mathrm{i}| \leqslant d\}$$

where a, b, c and d are integers.

(a) Write down the values of a, b, c and d.

(3)

The set Q is defined by

$$Q = \{z \in \mathbb{C} : a \leqslant |z + b + c\mathrm{i}| \leqslant d\} \cap \{z \in \mathbb{C} : |z - \mathrm{i}| \leqslant |z - 3\mathrm{i}|\}$$

(b) Determine the exact area of the region defined by Q, giving your answer in simplest form.

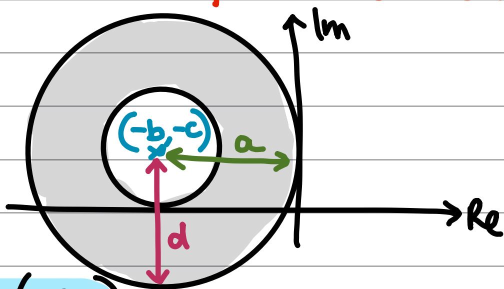
(7)

5. a) $|z-(x_1+iy_1)| = r$ is a circle on an Argand diagram: centre (x_1, y_1) radius r

because every point z, on the circumference of the circle is a distance of r from the centre of the circle

$z : a \leq |z+b+ci| \leq d$

$z : a \leq |z-(-b-ci)| \leq d$



$\therefore$ centre of 2 circles is $(-b,-c) = (-2,1)$

$\therefore b = 2 \quad c = -1$

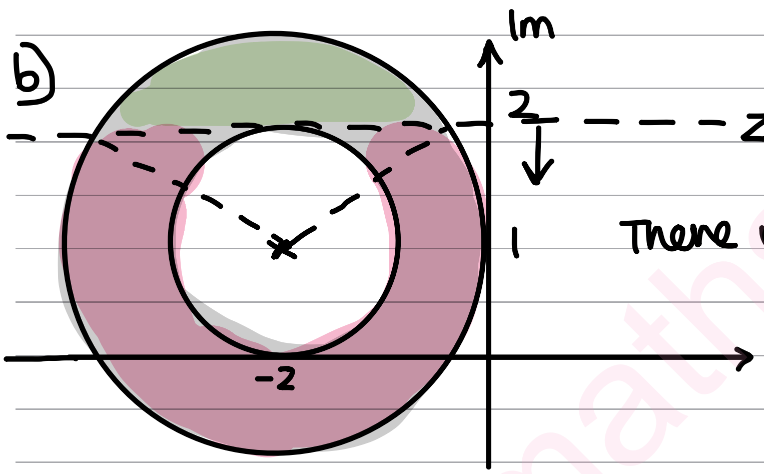
P 6 6 7 9 0 A 0 1 2 3 2

The inequality signs show that any point z, should be less than d but more than a away from the centre

∴ the larger radius $= d = 2$

smaller radius $= a = 1$

b)



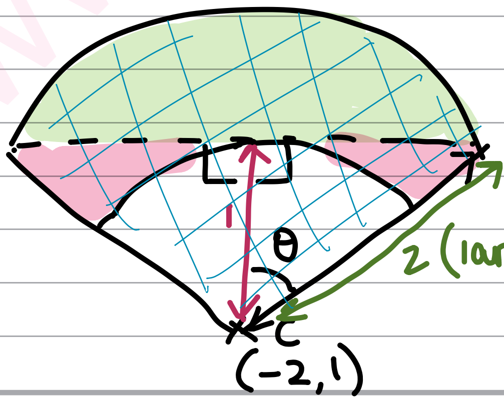
$z : |z-i| \leq |z-3i|$

There would be a perpendicular bisector between the points (0,1) and (0,3)
The section by the inequality is the side of the line closer to (0,1)
∴ $y \leq 2$

To find the area, first find the total area between the 2 circles

$\pi(2)^2 - \pi(1)^2 = 4\pi - \pi = 3\pi$

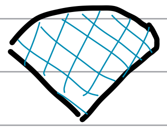
To find the top area to subtract



$\cos\theta = \frac{\text{adj.}}{\text{hypot.}} = \frac{1}{2}$

$\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60°$

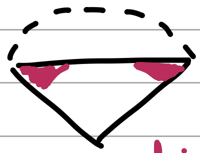
Question 5 continued



Total area of sector $= \pi r^2 \times \frac{2\theta}{360}$

$= \pi (2)^2 \times \frac{120}{360}$

$= \frac{4\pi}{3}$



Total area of triangle in bottom half of sector $= \frac{1}{2} \times a \times b \times \sin C$

$= \left(\frac{1}{2} \times 2 \times 1 \times \sin 60\right) \times 2$

$= \sqrt{3}$

half i.e right angled triangle

$\therefore$ Required area to subtract

$= \frac{4\pi}{3} - \sqrt{3}$

$\therefore Q = 3\pi - \left(\frac{4\pi}{3} - \sqrt{3}\right)$

$= \frac{5\pi}{3} + \sqrt{3}$

6. A mining company has identified a mineral layer below ground.

The mining company wishes to drill down to reach the mineral layer and models the situation as follows.

With respect to a fixed origin O,

- the ground is modelled as a horizontal plane with equation $z = 0$
- the mineral layer is modelled as part of the plane containing the points $A(10, 5, -50)$, $B(15, 30, -45)$ and $C(-5, 20, -60)$, where the units are in metres

(a) Determine an equation for the plane containing A, B and C, giving your answer in the form $\mathbf{r.n} = d$

(5)

(b) Determine, according to the model, the acute angle between the ground and the plane containing the mineral layer. Give your answer to the nearest degree.

(3)

The mining company plans to drill vertically downwards from the point (5, 12, 0) on the ground to reach the mineral layer.

(c) Using the model, determine, in metres to 1 decimal place, the distance the mining company will need to drill in order to reach the mineral layer.

(2)

(d) State a limitation of the assumption that the mineral layer can be modelled as a plane.

(1)

5.a) • Find 2 direction vectors in the plane

$$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} 15 \\ 30 \\ -45 \end{pmatrix} - \begin{pmatrix} 10 \\ 5 \\ -50 \end{pmatrix} = \begin{pmatrix} 5 \\ 25 \\ 5 \end{pmatrix} \Rightarrow k \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$$

$$\vec{AC} = \vec{OC} - \vec{OA} = \begin{pmatrix} -5 \\ 20 \\ -60 \end{pmatrix} - \begin{pmatrix} 10 \\ 5 \\ -50 \end{pmatrix} = \begin{pmatrix} -15 \\ 15 \\ -10 \end{pmatrix} \Rightarrow k \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}$$

• Calculate a normal to the plane from the 2 direction vectors

$a \times b = \text{normal}$

$$\vec{AB} \times \vec{AC} = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 13 \\ 1 \\ -18 \end{pmatrix}$$

P 6 6 7 9 0 A 0 1 6 3 2

Question 6 continued

equation of plane

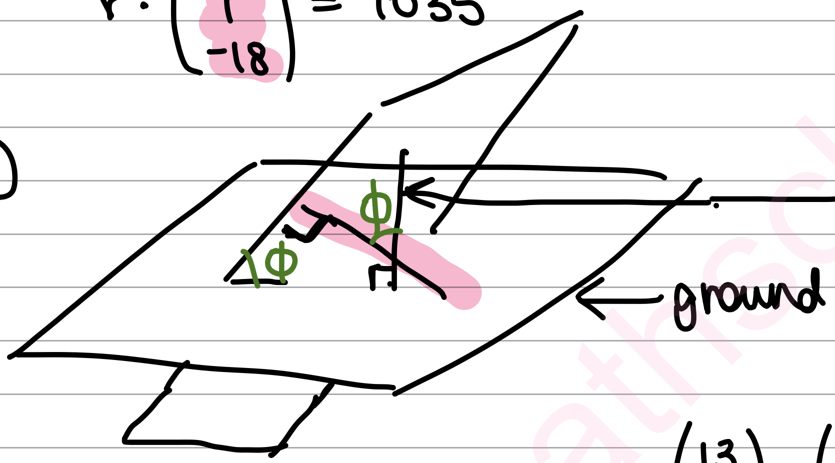
$r \cdot n = d \qquad r \cdot \begin{pmatrix} 13 \\ 1 \\ -18 \end{pmatrix} = d$

• Choose any point on the plane to use as r & calculate d

$A \cdot n = \begin{pmatrix} 10 \\ 5 \\ -50 \end{pmatrix} \cdot \begin{pmatrix} 13 \\ 1 \\ -18 \end{pmatrix} = 1035 = d$

$r \cdot \begin{pmatrix} 13 \\ 1 \\ -18 \end{pmatrix} = 1035$

b)



normal to the plane $z = 0$ is the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\boxed{\cos\phi = \frac{a \cdot b}{|a||b|}}$

$\cos\phi = \dfrac{\begin{pmatrix} 13 \\ 1 \\ -18 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} 13 \\ 1 \\ -18 \end{pmatrix}\right| \left|\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|} = \dfrac{-18}{\sqrt{494}}$

The angle between 2 normals of 2 planes $\phi = 144°$

$\hookrightarrow 180 - 144 = 36°$

= angle between 2 planes

c) Point directly below $\begin{pmatrix} 5 \\ 12 \\ 0 \end{pmatrix}$ on the plane must be of the form $\begin{pmatrix} 5 \\ 12 \\ -\lambda \end{pmatrix}$

The point must also fit on the plane:

$r \cdot \begin{pmatrix} 13 \\ 1 \\ -18 \end{pmatrix} = 1035$

Question 6 continued

$$\begin{pmatrix} 5 \\ 12 \\ \lambda \end{pmatrix} \cdot \begin{pmatrix} 13 \\ 1 \\ -18 \end{pmatrix} = 1035$$

$$65 + 12 - 18\lambda = 1035$$

$$\lambda = -\frac{479}{9}$$

$\therefore$ the distance travelled vertically down

$= |\lambda|$

$= 53.2\text{ m}$

d) The mineral layer won't be perfectly smooth, neither will it be completely flat, it would have a depth

7. $$f(z) = z^4 - 6z^3 + pz^2 + qz + r$$

where p, q and r are real constants.

The roots of the equation $f(z) = 0$ are α, β, γ and δ where $\alpha = 3$ and $\beta = 2 + i$

Given that γ is a complex root of $f(z) = 0$

(a) (i) write down the root γ,

(ii) explain why δ must be real.

(2)

(b) Determine the value of δ.

(2)

(c) Hence determine the values of p, q and r.

(3)

(d) Write down the roots of the equation $f(-2z) = 0$

(2)

7.a) (i) A polynomial with real coefficients will always have complex solutions in conjugates

i.e. if $a+bi$ is a solution
$a-bi$ is also a solution

$\therefore \gamma = (2+i)^* = 2-i$

(ii) Because 2 of the roots are already complex conjugates and we know that the 3rd root is real $\therefore \delta$ must also be real

b) $ax^4 + bx^3 + cx^2 + dx + e = 0$

sum of roots   sum of all possible products of 2 roots $\alpha\beta\gamma\delta$

$\Sigma \alpha_i = -\frac{b}{a}$ $\Sigma a_i a_j = \frac{c}{a}$ $\Sigma a_i a_j a_k = -\frac{d}{a}$ $\Sigma \alpha_i \alpha_j \alpha_k \alpha_l = \frac{e}{a}$

sum of all possible products of 3 roots

$\therefore \frac{6}{1} = \alpha + \beta + \gamma + \delta = 3 + 2 + i + 2 - i + \delta$

$6 = 7 + \delta$

$\delta = -1$

c) METHOD 1 (using sum/product rules of polynomial):

$z^4 - 6z^3 + pz^2 + qz + r$

$p = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

$= 6 + 3i + 6 - 3i - 3 + 5 - 2 - i - 2 + i$

$= 10$

$-q = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$

$= 15 - 6 - 3i - 6 + 3i - 5 = -2$

$q = 2$

$r = \alpha\beta\gamma\delta = -15$

METHOD 2 (expand brackets):

$f(z) = (z-3)\underbrace{(z-(2+i))(z-(2-i))}_{\text{complex conjugates}}(z+1)$

$= (z^2 - 2z - 3)(z^2 - 4z + 5)$

$= z^4 - 6z^3 + 10z^2 + 2z - 15$

d) replace z with $-2z$

$f(-2z) = (-2z-3)(-2z+1)(-2z-(2+i))(-2z-(2-i))$

$f(-2z) = 0 \qquad z = -\frac{3}{2} \cup \frac{1}{2} \cup -\left(\frac{2+i}{2}\right) \cup -\left(\frac{2-i}{2}\right)$

$z = -\frac{3}{2} \cup \frac{1}{2} \cup -1 - \frac{i}{2} \cup -1 + \frac{i}{2}$

8. (a) Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} r(r+1)(2r+1) = \frac{1}{2}n(n+1)^2(n+2)$$

(6)

(b) Hence, show that, for all positive integers n,

$$\sum_{r=n}^{2n} r(r+1)(2r+1) = \frac{1}{2}n(n+1)(an+b)(cn+d)$$

where a, b, c and d are integers to be determined.

(3)

8.a) INDUCTION:
- Prove true for base case
- Assume true for $n=k$
- consider $n=k+1$ & replace by assumption
- Conclusion

Base case $n=1$:

LHS: $\sum_{r=1}^{1} r(r+1)(2r+1) = 6$ RHS: $\frac{1}{2}(1)((1)+1)^2((1)+2)$

$= \frac{1}{2} \times 4 \times 3 = 6$

$\therefore$ statement true for $n=1$

Assume true for $n=k$

$$\sum_{r=1}^{k} r(r+1)(2r+1) = \frac{1}{2}k(k+1)^2(k+2)$$

Consider $n=k+1$

$$\sum_{r=1}^{k+1} r(r+1)(2r+1) = (k+1)((k+1)+1)(2(k+1)+1) + \sum_{r=1}^{k} r(r+1)(2r+1)$$

$$= (k+1)(k+2)(2k+3) + \frac{1}{2}k(k+1)^2(k+2)$$

$$= (k+1)(k+2)\left((2k+3) + \frac{k^2+k}{2}\right)$$

$$= \frac{(k+1)(k+2)}{2}(k^2+5k+6) = \frac{1}{2}(k+1)((k+1)+1)^2((k+1)+2)$$

$\therefore$ hence true for $n=k+1$

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Question 8 continued

Since statement is true for $n=1$, and we have proved if true for $n=k$, it is true for $n=k+1$, thus by mathematical induction, the result holds true for all positive integers

b) split summation

e.g. $\displaystyle\sum_{r=1}^{2n} r = \underbrace{1+2+\ldots(n-1)}_{\sum_{r=1}^{n-1} r} + \overbrace{n+(n+1)+\ldots(2n-1)+2n}^{\sum_{r=n}^{2n} r}$

(using identity proved in (a) $\displaystyle\sum_{r=1}^{n} r(r+1)(2r+1) = \frac{1}{2}n(n+1)^2(n+2)$)

$$\sum_{r=n}^{2n} r(r+1)(2r+1) = \sum_{r=1}^{2n} r(r+1)(2r+1) - \sum_{r=1}^{n-1} r(r+1)(2r+1)$$

$$= \frac{1}{2}(2n)((2n)+1)^2((2n)+2) - \frac{1}{2}(n-1)((n-1)+1)^2((n-1)+2)$$

$$= \frac{1}{2}\left(2n(2n+1)^2(2n+2) - (n-1)(n)^2(n+1)\right)$$

$$= \frac{1}{2}\left(2n(2n+1)^2(2(n+1)) - (n-1)n^2(n+1)\right)$$

$$= \frac{n(n+1)}{2}\left(4(2n+1)^2 - n(n-1)\right)$$

$$= \frac{n}{2}(n+1)\left(4(4n^2+8n+1) - n^2+n\right)$$

$$= \frac{n}{2}(n+1)\left(16n^2+16n+4-n^2+n\right)$$

$$= \frac{n}{2}(n+1)(15n^2+17n+4)$$

$$= \frac{n}{2}(n+1)(3n+1)(5n+4)$$

9.

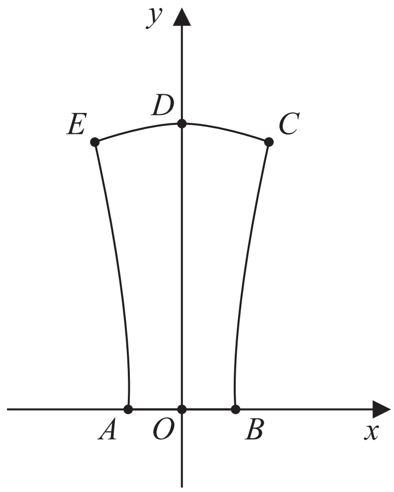


Figure 2

Figure 2 shows the vertical cross-section, $AOBCDE$, through the centre of a wax candle.

In a model, the candle is formed by rotating the region bounded by the y-axis, the line OB, the curve BC, and the curve CD through 360° about the y-axis.

The point B has coordinates (3, 0) and the point C has coordinates (5, 15).

The units are in centimetres.

The curve BC is represented by the equation

$$y = \frac{\sqrt{225x^2 - 2025}}{a} \qquad 3 \leqslant x < 5$$

where a is a constant.

(a) Determine the value of a according to this model.

(2)

The curve CD is represented by the equation

$$y = 16 - 0.04x^2 \qquad 0 \leqslant x < 5$$

(b) Using algebraic integration, determine, according to the model, the exact volume of wax that would be required to make the candle.

(9)

(c) State a limitation of the model.

(1)

When the candle was manufactured, 700 cm^3 of wax were required.

(d) Use this information and your answer to part (b) to evaluate the model, explaining your reasoning.

(1)

Question 9 continued

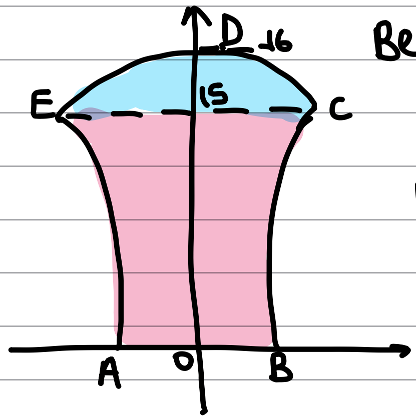
9. a) C : (5, 15)

$$y = 15 = \frac{\sqrt{225(5)^2 - 2025}}{a}$$

$a = 4$

b) Volume of curve rotated about y-axis :

$$V = \pi \int x^2 \, dy$$



Because we have 2 separate curves, we must calculate 2 parts separately

Have to write equations in terms of y

$\therefore$ BC : $y = \dfrac{\sqrt{225x^2 - 2025}}{4}$

$16y^2 = 225x^2 - 2025$

$x = \dfrac{\sqrt{16y^2 + 2025}}{15}$

CD : $y = 16 - 0.04x^2$

$x^2 = -25y + 400$

$x = \sqrt{400 - 25y}$

$$V_{BC} = \pi \int_0^{15} \left(\frac{\sqrt{16y^2 + 2025}}{15} \right)^2 dy = \frac{\pi}{225} \int_0^{15} 16y^2 + 2025 \, dy$$

$$= \frac{\pi}{225} \left[\frac{16}{3} y^3 + 2025y \right]_0^{15} = \frac{48375\pi}{225} = 215\pi$$

$$V_{CD} = \pi \int_{15}^{16} \left(\sqrt{400 - 25y} \right)^2 dy$$

Question 9 continued

$$= \pi \int_{15}^{16} 400 - 25y \, dy$$

$$= \pi \left[400y - \frac{25}{2}y^2 \right]_{15}^{16}$$

$$= \pi \left(3200 - \frac{6375}{2} \right)$$

$$= \frac{25\pi}{2}$$

Total volume $= 215\pi + \frac{25\pi}{2} = \frac{455}{2}\pi \text{ cm}^3$ $(\approx 715 \text{ cm}^3)$

c) - The sides of the candle won't be perfectly smooth
- May be a whole in the middle for a wick
- Equations for curve may not be entirely suitable

d) 715 is only 15cm^3 more than the actual wax used (700) $\therefore$ it is a good estimate.

(Total for Question 9 is 13 marks)

TOTAL FOR CORE PURE MATHEMATICS IS 80 MARKS